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Kink dynamics in the periodically modulated ϕ^4 model

Zhang Fei
Departamento de Física Teórica I, Facultad de Ciencias Físicas, Universidad Complutense, E-28040 Madrid, Spain and Research School of Chemistry, Australian National University, GPO Box 4, Canberra, Australia Capital Territory, Australia 2601

Vladimir V. Konotop*
Departamento de Física Teórica I, Facultad de Ciencias Físicas, Universidad Complutense, E-28040 Madrid, Spain

Michel Peyrard
Laboratoire de Physique, Ecole Normale Supérieure, 46 Allée d'Italie, 69364 Lyon CEDEX 07, France

Luis Vázquez
Departamento de Física Teórica I, Facultad de Ciencias Físicas, Universidad Complutense, E-28040 Madrid, Spain
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Kink dynamics in the spatially modulated ϕ^4 model is studied analytically with the Born approximation and a collective-coordinate approach, and numerically. Four types of kink behavior can be distinguished: (i) radiation at high velocity, (ii) strong resonant beating in the kink velocity, (iii) propagation of the kink with almost periodic velocity oscillations, and (iv) trapping at low velocity. Although some of these regimes exist also in the sine-Gordon (SG) case, the ϕ^4 model exhibits a much richer dynamics due to the existence of a localized internal mode of the kink, which can exchange energy with the translational mode. The propagation in a modulated substrate is very sensitive to this effect, and in this respect the ϕ^4 model could be more representative of real systems which may not have the peculiar properties of the integrable SG model.

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I. INTRODUCTION

Kinks in one-dimensional systems have been used to describe various phenomena in solid-state physics such as dislocations [1–5], ferroelectric [6,7] or ferromagnetic domain walls [8,9], or charge-density waves [10]. The basic models consider a set of harmonically coupled particles (or an elastic string, in the continuum limit) subjected to a substrate potential which is either periodic (like in the sine-Gordon model) or has a double-well shape (like in the ϕ^4 model). The parameters of this substrate potential are generally assumed to be independent of space; however, there are many physical systems in which this assumption does not hold. One example is the case of disordered systems [11] which is currently under active investigation. Modulated structure, which can be found, for instance, in two- or three-dimensional incommensurate phases [12] or ferroelastic materials [13], and in alloys undergoing a spinodal decomposition [14], form an important class of examples in which the substrate potential is modulated periodically. Another case which is closely connected to this problem is the propagation of a kink in a discrete lattice because discreteness effects create an additional "Peierls-Nabarro potential" that adds a short-wavelength modulation to the substrate [15].

The properties of the sine-Gordon (SG) model perturbed by a spatial modulation of wave vector κ of the substrate potential have been investigated recently [16–18]. It was found that a moving kink radiates linear waves when its velocity is larger than a critical value, $V_{\rm cr}=(1+\kappa^2)^{-1/2}$, in analogy with the properties of a kink in a discrete lattice. The dynamics of a breather mode in the modulated SG model has also been analyzed in Refs. [19,20].

The purpose of the present work is to investigate the properties of kinks in the modulated ϕ^4 model:

$$\phi_{tt}-\phi_{xx}+[1+\epsilon\sin(\kappa x)](\phi^3-\phi)=0 , \quad (1)$$

where $|\epsilon| \ll 1$. Our aim is not simply to add a new particular case to the previous studies. It is more fundamental because the SG model that was treated is very peculiar and is not representative of the situation that one may expect in most of the modulated physical systems. The complete integrability of the SG model is appealing for the analysis, but, in this system, the spectrum of the small excitations around the soliton is restricted to the zero-frequency Goldstone mode and the nonlocalized phonon excitations corresponding to the continuum part. This spectrum is obtained by solving a Schrödinger-like equation for the deviation around the kink, and, in the Klein-Gordon model bearing topological kinks, the potential created by the kink has the shape of a well. As soon as this well is deep enough, one can expect several localized modes and such a situation indeed occurs in many models such as deformable sine-Gordon models [21], the double sine-Gordon model [22], or the ϕ^4 model that we consider here [23]. Therefore, when one considers oscillations around the kink, the SG model appears to

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be the exception rather than the rule. But for the problem that we investigate here, the spectrum of the small oscillations around the kink is crucial because the spatial modulation of the parameters creates a periodic perturbation of the moving kink, which excites small oscillations that can take energy away from the kink. The ϕ^4 model is interesting because it is representative of a larger class of physical models than the SG model, and the spectrum of its small-amplitude oscillations around the kink is well known. The ϕ^4 kink has an internal shape mode which can be easily excited due to kink collisions [23] or kink-impurity interactions [24]. In these cases, energy exchange between the kink translational mode and the internal mode gives rise to spectacular resonance effects in the kink dynamics [23,24]. It is the purpose of the present paper to show that the existence of such an internal mode causes *qualitative* changes on the dynamics of the kink in the modulated system with respect to the SG case.

Although the original unperturbed ϕ^4 system is not exactly integrable (in contrast to the SG model), the main effects of the modulation can be predicted analytically due to the well-known mode structure of the corresponding linearized problem (see Ref. [25] and references therein). The velocity range in which the kink radiates phonons can be investigated using either the Born approximation [26] or the secular perturbation theory [27,28]. This aspect is presented in Sec. II. Having determined the threshold velocity under which radiations do not appear, we use a collective-coordinate approach to analyze the low-velocity domain in which the dynamics is strongly affected by the internal mode (Sec. III). Section IV presents numerical simulations used to check the analytical results, and Sec. V discusses the results.

II. THEORETICAL ANALYSIS IN THE BORN APPROXIMATION

Since we are interested in the kink dynamics, it is natural to use a collective-coordinate approach. However, the previous studies of the modulated sine-Gordon model have exhibited a high-velocity domain in which the kink radiates phonons permanently, thus precluding an approach considering only the localized aspect of the kink. Therefore, it is necessary to start the analysis of the ϕ^4 model with a method which is suitable to treat radiation. This is the case of the Born approximation [26] which looks for a solution in the form of a perturbed kink:

$$\phi(x,t)=\phi_k(x,t)+\psi(x,t)\ , \tag{2}$$

where

$$\phi_k(x,t)=\tanh[\gamma(x-Vt-x_0)]\ ,\quad \text{with } \gamma=\frac{1}{[2(1-V^2)]^{1/2}}\ , \tag{3}$$

is the exact kink solution of the unperturbed ϕ^4 equation and ψ is assumed to be small ($|\psi|\ll 1$). Unlike the secular perturbation theory, the Born approximation does not take into account a slow time dependence of the kink parameters, so it can only describe small deviations around the unperturbed solution, and it is usually only valid within a restricted time interval which can be roughly estimated as $t\ll\epsilon^{-1}$. However, it has the advantage of giving an evaluation of the correction $\psi(x,t)$ in a closed form (at least as integrals).

Substituting Eqs. (2) and (3) into (1) and keeping terms of the first order in ϵ, we obtain a linear equation,

$$\psi_{tt}-\psi_{xx}+(3\phi_k^2-1)\psi=-\epsilon\sin(\kappa x)(\phi_k^3-\phi_k)\ , \tag{4}$$

which can be solved analytically since the eigenvalue problem of the linear operator,

$$L_{\phi^4}\equiv-\partial_{xx}+(3\phi_k^2-1)\ ,$$

is known [25]. Four our purpose here, it is more convenient to analyze the mode structure directly rather than the final expression of the Green's function. In this section we follow the approach developed in Ref. [28], keeping the same notations.

Defining new coordinates (t,ζ), $\zeta=\gamma(x-Vt-x_0)$, instead of (t,x), the solution of Eq. (4), $\psi(t,\zeta)$, is decomposed into three contributions: the translational mode $\phi^{(0)}(t,\zeta)$, the kink internal mode $\phi^{(1)}(t,\zeta)$, and the phonon modes (or radiation) $\phi^{(\mathrm{ph})}(t,\zeta)$,

$$\psi(t,\zeta)=\phi^{(0)}(t,\zeta)+\phi^{(1)}(t,\zeta)+\phi^{(\mathrm{ph})}(t,\zeta)\ . \tag{5}$$

After a tedious manipulation [30], the three contributions can be evaluated analytically. The phonon contribution determines whether the kink radiates or not. It is given by

$$\phi^{(\mathrm{ph})}(\zeta,t)=\int_{-\infty}^{\infty}dk[A_-(k;\zeta,t)e^{iK_-\zeta+i\Omega_k t}+A_+(k;\zeta,t)e^{iK_+\zeta-i\Omega_k t}]\ , \tag{6}$$

where

$$A_\pm(k;\zeta,t)=\pm\frac{1-V^2}{2\pi}\frac{3\tanh^2\zeta-3ik\tanh\zeta-(1+k^2)}{(1+k^2)(4+k^2)}\left[\frac{\Omega_k\exp[\pm i(\kappa V-\Omega_k)(t-2V\gamma\zeta)]}{\Omega_k^2-(\kappa V)^2}I^\pm(2\gamma\kappa,k)+\frac{1}{2(\Omega_k+\kappa V)}I^\pm[\alpha_-(\Omega_k),k]-\frac{1}{2(\Omega_k-\kappa V)}I^\pm[\alpha_+(\Omega_k),k]\right]\ , \tag{7}$$

$$I^\pm(\alpha,k)=\nu(\alpha\pm k)g(\alpha;\pm k),\quad g(\alpha;k)=-\alpha^3+4\alpha+\alpha k^2-\alpha^2k-4k-k^3\ , \tag{8}$$

$$K_\pm=k\pm V(k^2+4)^{1/2}\ , \tag{9}$$

$$\Omega_k^2=(k^2+4)(2\gamma)^{-2}\ . \tag{10}$$

Its main qualitative features can be found from a simple study of the poles of the integrand of (6). Let us consider, for instance $A_{-}(k,\zeta)$ (A_{+} is similar). We find the following set of poles:

$$k_1^{\pm}=\pm i,\quad k_2^{\pm}=\pm 2i\ , \tag{11a}$$

$$k_n^{\pm}=\frac{\gamma}{4V}\{1+[1-8V(8V\gamma^{-2}\pm\kappa\gamma^{-2}-2in\gamma^{-1})]^{1/2}\}\ , \tag{11b}$$

$$\tilde{k}_n^{\pm}=\frac{\gamma}{4V}\{1-[1-8V(8V\gamma^{-2}\pm\kappa\gamma^{-2}-2in\gamma^{-1})]^{1/2}\}\ , \tag{11c}$$

$$k_0^{\pm}=\pm 2(\gamma^2\kappa^2V^2-1)^{1/2}\ . \tag{11d}$$

The first four poles (11a) are generated by the factor $1/(k^2+1)(k^2+4)$. The sets $k_n^{\pm}$ and $\tilde{k}_n^{\pm}$ (with n being a nonzero integer) result from the functions $\nu(\ )$. The last two poles (11d) are related to the denominators containing Ω_k. It follows directly that if $V<V_c^{(\mathrm{ph})}$, where

$$V_c^{(\mathrm{ph})}=\frac{1}{(1+\kappa^2/2)^{1/2}}\ , \tag{12}$$

all the poles have imaginary parts; thus the corresponding solution $\phi^{(\mathrm{ph})}(\zeta,t)$ is localized, i.e., it decays exponentially with $|\zeta|$. On the contrary, if $V>V_c^{(\mathrm{ph})}$, the poles $k_0^{\pm}$ are purely real, which means that an extended component appears in the fields ψ. Therefore, $V_c^{(\mathrm{ph})}$ is a critical velocity that separates a high-velocity domain in which the kink emits radiation from a low-velocity domain in which the kink propagates without radiative losses. The phase velocity of the radiation near the threshold point is $1/V_c^{(\mathrm{ph})}$. The structure of the radiation ahead of and behind the kink is described by $A_{+}(k_0^{\pm};\zeta,t)$ and $A_{-}(k_0^{\pm};\zeta,t)$, [i.e., by the contribution of the poles (11d)], respectively. If the kink velocity approaches unity, the radiation amplitude goes to zero in accordance with a power law behind the kink [due to the prefactor $(1-V^2)$ in expression (7)] and an exponential law ahead of the kink [because of the factor $\nu(\alpha+k)$]. These results for the phonon modes are analogous to those found in the SG system under the same type of perturbation [16,17].

However, an essential difference with the SG case arises in the ϕ^4 case due to the existence of the kink internal mode, which gives rise to the contribution

$$\phi^{(1)}(t,\zeta)=-\frac{\pi\Omega}{120}\frac{\sinh\zeta}{\cosh^2\zeta}\times\left[\frac{\Omega}{\Omega^2-(\kappa V)^2}\chi[2\gamma\kappa,\beta(\kappa V)]+\frac{\chi[\alpha_{-}(\Omega),\beta(\Omega)]}{2(\Omega+\kappa V)}-\frac{\chi[\alpha_{-}(\Omega),\beta(\Omega)]}{2(\Omega-\kappa V)}\right]\ , \tag{13}$$

where

$$\Omega=[\tfrac{3}{2}(1-V^2)]^{1/2} \tag{14}$$

is the frequency of the internal mode of a moving kink and

$$\chi(\alpha,\beta)=\frac{\sin\beta}{\cosh(\alpha\pi/2)}(\alpha^2+1)(11-\alpha^2)\ , \tag{15}$$

$$\beta(\omega)=\omega(t-2V\gamma\zeta)\ , \tag{16}$$

$$\alpha_{\pm}(\omega)=2V\gamma\omega\pm\gamma^{-1}\kappa\ . \tag{17}$$

The amplitude of the internal mode $\phi^{(1)}$ has a pole at velocity

$$V=V^{(1)}\equiv\frac{1}{(1+2\kappa^2/3)^{1/2}}\ , \tag{18}$$

and, taking the limit $V\to V^{(1)}$, we find that it contains a term which grows linearly with time. Although this result is unphysical because it comes from the assumption of a constant kink velocity in the perturbation method, it shows that the internal mode can be strongly excited if the kink velocity approaches the value given by (18).

It is interesting to notice that the calculation does not yield a secular growth for the translation mode of the kink, $\phi^{(0)}$ (contrary to the case of a stochastic perturbation [28]) because, here, the modulation breaks the translational invariance of the system.

III. COLLECTIVE-COORDINATE APPROACH

The Born approximation fails to describe properly the long-time contribution of the internal mode to the kink dynamics in the modulated structure, but it shows that the characteristic velocity at which the internal mode resonates is smaller than the critical velocity above which the kink radiates phonons. Therefore, the role of the internal mode can be investigated in the velocity range

$$V<V_c^{(\mathrm{ph})}=(1+\kappa^2/2)^{-1/2}$$

with a collective-coordinate method that neglects the phonon modes.

Assuming that the kink is moving with an average velocity V, we consider Eq. (1) in the moving frame defined by the Lorentz transformation

$$z=\frac{(x-Vt)}{(1-V^2)^{1/2}},\quad \tau=\frac{(t-Vx)}{(1-V^2)^{1/2}}\ . \tag{19}$$

It becomes

$$\phi_{\tau\tau}-\phi_{zz}+\{1+\epsilon\sin[\bar{\kappa}(z+V\tau)]\}(\phi^3-\phi)=0\ , \tag{20}$$

where $\bar{\kappa}=\kappa(1-V^2)^{1/2}$.

Since the Born approximation has exhibited a secular growth of the internal mode, we introduce an additional free parameter $A(\tau)$, which can be adjusted to compensate for the growth in the spirit of the secular perturbation theory. However, this term alone would give birth to secular terms related to the zero mode, as shown in other cases [27,28]. The latter growth can be excluded by introducing slowly varying kink parameters. Moreover, since the internal mode is described by an odd function of z, it is sufficient to introduce a single adiabatic parameter describing the position of the kink center, without an additional correction for the width. Therefore, we look for a "wobbling" kink of the form

$$\phi(z,\tau)=\tanh[z-z_0(\tau)] + A(\tau)\left[\frac{9}{8}\right]^{1/4}\tanh[z-z_0(\tau)]\cosh^{-1}[z-z_0(\tau)] \; . \tag{21}$$

The equations for $z_0(\tau)$ and $A(\tau)$ are obtained through the effective Lagrangian approach. Inserting the ansatz (21) into the Lagrangian of the system

$$L=\int_{-\infty}^{\infty} dz\left(\tfrac{1}{2}\phi_\tau^2-\tfrac{1}{2}\phi_z^2-\tfrac{1}{4}\{1+\epsilon\sin[\bar{\kappa}(z+V\tau)]\}(\phi^2-1)^2\right) \tag{22}$$

and assuming that the time derivatives of the slowly varying parameters are of order ϵ, we obtain, in the second-order approximation, the effective Lagrangian

$$\mathcal{L}_e=\tfrac{1}{2}M_k\dot{z}_0^2+\tfrac{1}{2}(\dot{A}^2-\omega_1^2A^2) -\epsilon[U(\bar{\kappa})+AF(\bar{\kappa})]\sin[\bar{\kappa}(z_0+V\tau)] \; , \tag{23}$$

where

$$U(\bar{\kappa})=\frac{\bar{\kappa}\pi(\bar{\kappa}^2+2)}{6\sinh(\bar{\kappa}\pi/\sqrt{2})} \; , \tag{24}$$

$$F(\bar{\kappa})=\left[\frac{9}{2}\right]^{1/4}\frac{\pi(\bar{\kappa}^2+\frac{1}{2})(\bar{\kappa}^2-\omega_1^2)}{6\cosh(\bar{\kappa}\pi/\sqrt{2})} \; , \tag{25}$$

$$M_k=2\sqrt{2}/3, \quad \omega_1=\sqrt{\tfrac{3}{2}} \; . \tag{26}$$

The equation of motion for the collective coordinates $z_0(\tau)$ and $A(\tau)$ can be derived easily:

$$M_k\ddot{z}_0+\epsilon[U(\bar{\kappa})+AF(\bar{\kappa})]\bar{\kappa}\cos[\bar{\kappa}(z_0+V\tau)]=0 \; , \tag{27a}$$

$$\ddot{A}+\omega_1^2A+\epsilon F(\bar{\kappa})\sin[\bar{\kappa}(z_0+V\tau)]=0 \; , \tag{27b}$$

where the "dot" refers to the derivative with respect to τ.

The system (27a) and (27b) shows several important features of the kink dynamics. A first insight can be obtained by neglecting the internal-mode amplitude in Eq. (27a). This approximation decouples the two equations and allows an analytical treatment of the system (27a) and (27b). If we introduce the kink coordinate $X(\tau)=V\tau+z_0(\tau)$, the differential equation for X is simply the equation of motion of a particle in a periodic potential:

$$M_k\ddot{X}+\epsilon U(\bar{\kappa})\bar{\kappa}\cos(\bar{\kappa}X)=0 \; . \tag{28}$$

It shows that the kink can propagate in the modulated medium only if its maximal kinetic energy exceeds the height of the potential, which requires

$$V_{\max}^2\geq\frac{\epsilon\bar{\kappa}\pi(\bar{\kappa}^2+2)}{\sqrt{2}\sinh(\bar{\kappa}\pi/\sqrt{2})} \; . \tag{29}$$

Thus the *average* velocity of the kink must be larger than

$$V_a\approx\frac{V_{\max}}{2}=\frac{1}{2}\left[\frac{\epsilon\bar{\kappa}\pi(\bar{\kappa}^2+2)}{\sqrt{2}\sinh(\bar{\kappa}\pi/\sqrt{2})}\right]^{1/2} \; . \tag{30}$$

For a given wave vector κ of the modulation, Eq. (30) defines the velocity $V_a(\kappa)$ below which the kink will be trapped by the modulation. This effect is very similar to the trapping by discreteness effects when a narrow kink is launched in a lattice. A basic problem is to calculate the shape of the static kink in the modulated system. When the characteristic length of the modulation (wavelength $2\pi/\kappa$) is larger than the kink width, a perturbative expansion can be performed, and when the modulation length is short with respect to the kink width (i.e., $\kappa\gg 1$), it is possible to introduce an average kink using an idea of Kapitza [29]. Therefore the determination of the structure of the static trapped kink can be solved analytically in these two limits.

For $V>V_a(\kappa)$, substituting $X=V\tau$ into Eq. (27b) which gives the internal-mode amplitude, we get the equation of motion of a harmonic oscillator subjected to a periodic driving force whose frequency is $\kappa V(1-V^2)^{1/2}$. Thus, if the kink velocity is such that this frequency approaches the internal-mode frequency $\omega_1=\sqrt{\frac{3}{2}}$, a strong excitation of the internal mode will be observed. The characteristic velocity for this resonance is equal to the velocity $V^{(1)}(\kappa)$ found via the Born approximation [see Eq. (18)]. In addition, a measure of the degree of excitation of the internal mode as a function of the wave vector of the modulation is given by the factor $\epsilon F(\kappa)$, which measures the strength of the driving force applied to the internal-mode oscillator. Putting $\bar{\kappa}=(\kappa^2+\frac{1}{2})^{1/2}$ in Eq. (25), we get the strength of the driving force at the resonance velocity $V^{(1)}$,

$$f(\kappa)=\left[\frac{9}{2}\right]^{1/4}\frac{\epsilon\pi(\kappa^2+2)\kappa^2}{6\cosh[\pi(\kappa^2+\frac{3}{2})^{1/2}/\sqrt{2}]} \; . \tag{31}$$

The function $f(\kappa)$ is presented in Fig. 1, which shows that the excitation of the internal mode reaches a maximum for $\kappa\sim 3$. When $\kappa>6$, $f(\kappa)\approx 0$, so that the internal mode will not be significantly excited. This is also in agreement with the outcome of the Born approximation. Indeed, recalling Eq. (2.7), one easily finds that the amplitude of the internal mode decays exponentially when $\kappa\to\infty$.

A more complete treatment of the system (27a) and (27b) can be achieved by numerical simulation of the

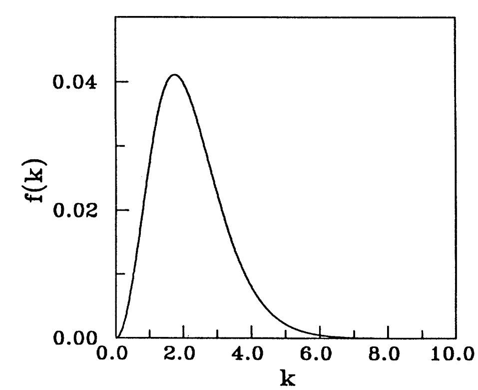

FIG. 1. The function $f(\kappa)$ defined by Eq. (31), giving the strength of the excitation of the internal mode as a function of the modulation wave vector.

collective-coordinate equations with initial conditions $z_0(0)=0$, $\dot{z}_0(0)=0$. Once $z_0(\tau)$ has been obtained, the position $X_0(t)$ of the kink center of mass in the original (x,t) frame can be obtained by reversing the Lorentz transform:

$$z-z_0(\tau)\equiv\frac{X_0-Vt}{(1-V^2)^{1/2}}-z_0\left[\frac{t-VX_0}{(1-V^2)^{1/2}}\right]\equiv 0\ . \qquad (32)$$

The simulations have been performed with a fixed $\epsilon=0.2$ and different values of κ. The average velocity V has been varied from 0 to $V_c^{(\mathrm{ph})}(\kappa)$. Some results are shown in Fig. 2. As expected from the simplified treatment, if V is below the threshold (30), the kink is trapped in a well of the modulation [see Fig. 2(a)]. For larger V, it can propagate and its velocity oscillates periodically [Fig. 2(b)]. When V approaches the resonance value $V^{(1)}$, the structure of the oscillations becomes more complex and *a beating* is observed [see Fig. 2(c)]. This beating phenomenon is due to an energy exchange between the kink translational mode and its internal shape mode. If the velocity approaches $V^{(1)}$, a significant amount of energy is transferred to the internal mode. Since the total energy of the system is conserved, the energy of the internal mode is taken away from the kinetic energy, causing a decrease in the kink velocity. In turn, this decrease reduces the energy transfer to the internal mode, which gets progressively deexcited, causing an increase of kinetic energy, and the process repeats again. Since the driving force of the internal mode depends on κ, as shown by Eq. (31), the beating is strong only if κ is not too large ($\kappa<6.0$).

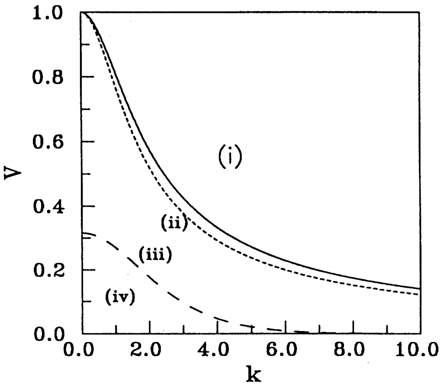

FIG. 3. Four regions in the phase plane (κ, V) where different kinds of kink dynamics may be observed: (i) radiative decay for high-velocity kinks, (ii) beating near the resonance velocity $V^{(1)}(\kappa)$, (iii) periodic oscillations of the kink velocity, and (iv) trapping for low-velocity kinks.

Based on the above analysis, we can derive general conclusions about the kink behavior in the modulated ϕ^4 model. They are summarized in Fig. 3: The (κ, V) plane can be divided into four regions characterized by different behaviors: (i) radiative decay for high-velocity kinks, (ii) beating near the resonance velocity $V^{(1)}(\kappa)$, (iii) periodic oscillations of the kink velocity, and (iv) trapping for low-velocity kinks. It should be noted that, while the limiting velocities for phonon radiation or trapping are sharply defined, the boundary between regions (ii) and (iii) is not well defined because the excitation of the internal mode which causes the beating increases *gradually* when $V\rightarrow V^{(1)}$.

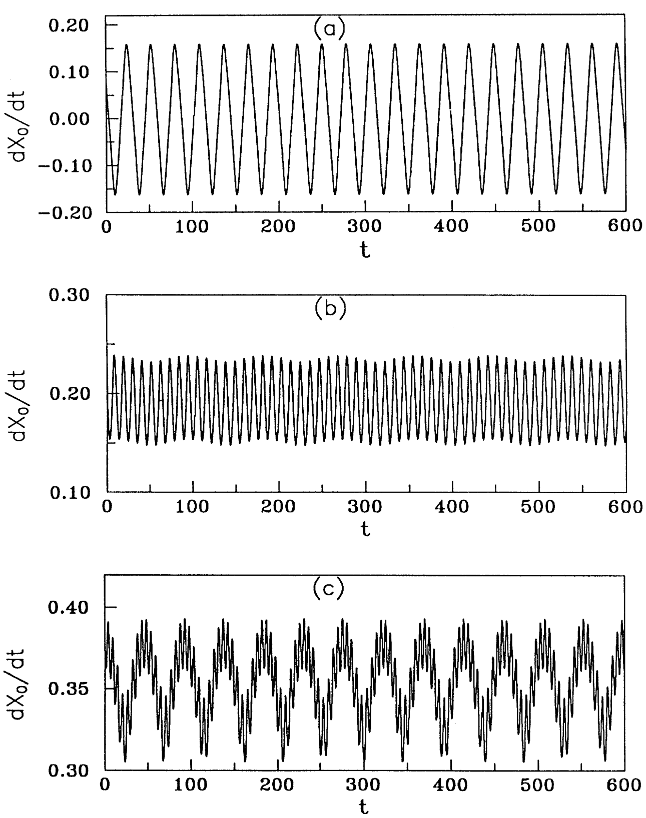

FIG. 2. Numerical simulation of Eqs. (27a) and (27b) with $\epsilon=0.2$ and $\kappa=3.0$. dX_0/dt defined by Eq. (32) is presented as function of time t. (a) Trapping of kink at $V=0.08$. (b) Periodic oscillation of kink velocity around $V=0.2$. (c) Beating near the resonant velocity $V=0.383$.

IV. NUMERICAL-SIMULATION RESULTS

Since all our analytical results are based on some approximation, it is important to verify them by direct numerical simulations of the full partial differential equation (PDE) (1). This has been done by discretizing the modulated ϕ^4 equation (1) with a finite-difference scheme which conserves a discrete energy [31,32]. The initial conditions are given by the kink solution of the unperturbed ϕ^4 model, and the kink is centered initially at $X_0=2n\pi/\kappa$ (n is a suitable negative integer). Most of the simulations are carried out in the spatial interval $(-120,120)$ up to time $T\leq 600$. For a given set of parameters ϵ and κ, we change the kink velocity V_i and observe the kink dynamics.

Fixing the strength of the perturbation $\epsilon=0.2$, we have carried out extensive numerical simulations for different values of κ. The main results are in very good agreement with the analytical predictions.

First, the critical velocity for radiation is easily observed by starting the kink at high velocity and following

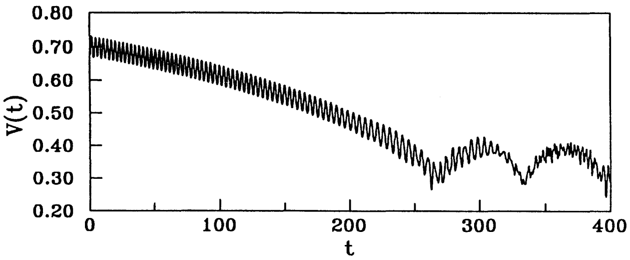

FIG. 4. Time evolution of the velocity of a kink launched with a high velocity in the modulated system ($\kappa=3.0$). The decay is due to radiative losses.

its evolution. Due to strong radiation, the kink velocity decreases fast until it drops below the critical value $V_c^{(\mathrm{ph})}$ given by Eq. (12). "Knees" are observed in Fig. 4 that present the kink velocity versus time. This is very similar to the case of kinks in discrete lattices [33,34]. Furthermore, we have checked the formula (12) for a number of the other values of κ, and a good agreement is observed. For instance, at $\kappa=3.0$, the critical velocity of radiation given by formula (12) is 0.426, while we notice in Fig. 4 that the final kink velocity (after the transition) is bounded by 0.43. In addition, the simulation shows that an ultrarelativistic kink with $V\sim 1$ decays slowly in agreement with the factor $(1-V^2)$ in the phonon amplitude [cf. Eq. (7)].

The beating phenomenon due to the internal mode is also clearly observed. For example, for $\kappa=3$, the resonant velocity given by Eq. (18) is $V^{(1)}(\kappa=3)=0.378$, which is chosen to be the initial velocity in the simulation shown in Fig. 5(a). We see that the velocity of the kink oscillates between 0.33 and 0.40, i.e., around the resonant value. For $\kappa=5$, the resonant value is $V^{(1)}(\kappa=2)=0.238$, and in Fig. 5(b) we indeed observe strong beating near

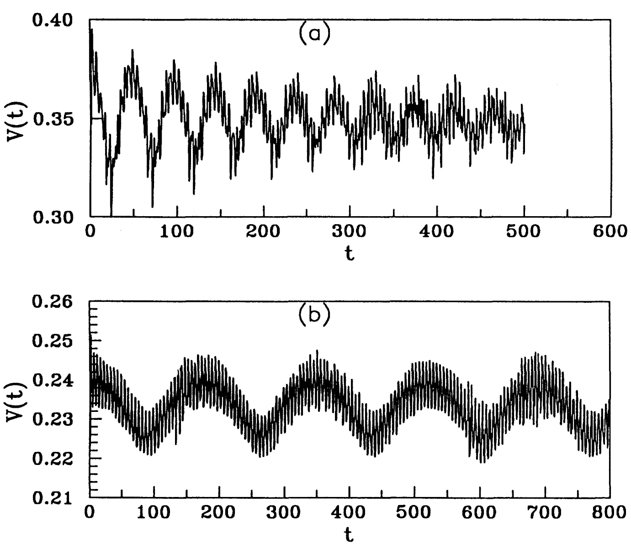

FIG. 5. Beating of the kink velocity for a kink propagating with a speed close to the resonant velocity $V^{(1)}(\kappa)$: (a) $\kappa=3.0$ and (b) $\kappa=5.0$.

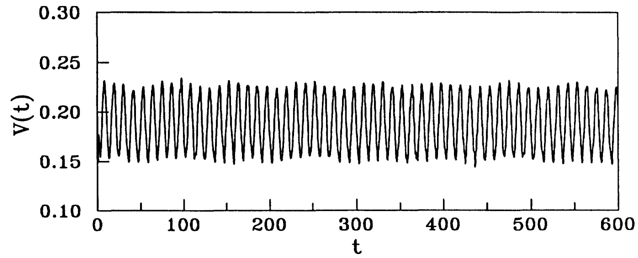

FIG. 6. Periodic oscillation of the kink velocity for a kink with an average velocity far away from the resonant value.

this velocity. Furthermore, we find that the beating is not present when κ is too large.

Third, for a lower velocity, farther away from $V^{(1)}$, a periodic oscillation of the kink velocity is observed while the kink is propagating along the system (Fig. 6). In this case the kink behaves like a point particle moving in a periodic potential well created by the perturbation. Its internal mode is not excited significantly since the periodic perturbation caused by the modulation is too far from the resonance condition.

Finally, the trapping of low-velocity kinks is also observed as shown in Fig. 7. The critical velocity obtained analytically within the collective-coordinate formalism is verified. For example, for $\kappa=\pi$, formula (30) gives $V_a=0.088$, while the simulation result is $V_a^{\mathrm{tr}}\approx 0.087$. At $\kappa=3\pi/2$, the analytical value is 0.027 and the numerical value is about 0.031.

V. CONCLUDING REMARKS

Our analytical and numerical study of the dynamics of a ϕ^4 kink in a periodically modulated substrate potential has revealed several interesting features. First, we have confirmed the existence of a critical velocity above which the kink radiates phonons. This phenomenon, which was also found analytically in the SG case [16–18], is not related to a specific model but it is a fundamental property of the nonlinear Klein-Gordon systems. Second, the kink dynamics is found to be richer in the modulated ϕ^4 model than in the SG due to the influence of the kink internal mode. Near a specific velocity, the ϕ^4 kink internal mode can be strongly excited and deexcited, creating a regular beating in the kink velocity. The fact that the internal mode can affect the kink dynamics drastically by its abili-

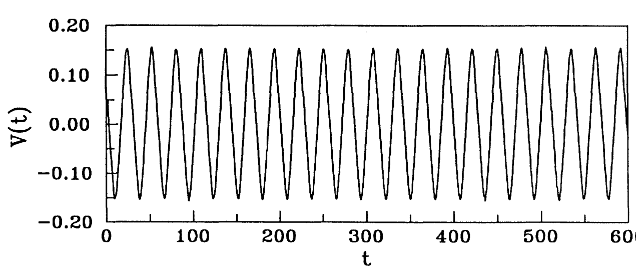

FIG. 7. Trapping of a kink launched with a low initial velocity. It is vibrating around the minimum of the effective potential due to the modulation.

ty to store and release significant amounts of energy was already known for kink-antikink collisions [23] and kink-impurity interactions [24], and the propagation over a modulated structure shows that this is a very general property.

Based on our results, we can expect to observe similar resonant beating effects in the other modulated KG models in which the kinks have an internal degree of freedom such as the double sine-Gordon [22] or the parametrized SG [21] models. As for the SG case, its kink has no true internal mode, so we would expect that the beating should not appear in such a model. Nevertheless, the existence of the so-called "quasimode" of the SG kink [35], which is a long-lived oscillation of the kink shape, may also create some kinds of beating effects. Recent preliminary numerical simulations [36] show that the beating effects in the SG model are not as well defined as those in the ϕ^4 model.

Finally, we would like to mention two points. First, as previously noted there is a strong similarity between the case of a modulated substrate and the properties of kinks in discrete lattices. Therefore, the analytical calculations that we have performed here can be expected to be valid to predict some properties of discrete lattices, as long as the amplitude of the Peierls-Nabarro potential has been calculated. Second, the numerical simulations of the full PDE have shown a remarkable agreement with the analytical calculations. In particular, our collective-coordinate approach has successfully predicted the resonant beating effects in the kink velocity. This points out once again that, while topological solitons (kinks) can behave like classical "particles," the internal degrees of freedom of these "particles," when they exist, must be taken into account in the collective-coordinate analysis.

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*Permanent address: Institute for Radiophysics and Electronics, 12 Proscura Street, 310085 Kharkov, Ukraine. Current address: Universidad da Madeira, Colegio dos Jesuitas, Largo do Municipio, 9000 Funchal, Portugal.

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